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THE LINEAR SYSTEM THEORY'S ACCOUNT OF BEHAVIOR MAINTAINED BY VARIABLE-RATIO SCHEDULES

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The mathematical theory of linear systems, which has been used successfully to describe behavior maintained by variable-interval schedules, is extended to describe behavior maintained by variable-ratio schedules. The result of the analysis is a pair of equations, one of which expresses response rate on a variable-ratio schedule as a function of the mean ratio requirement ($\bar{n}$) that the schedule arranges. The other equation expresses response rate on a variable-ratio schedule as a function of reinforcement rate. Both equations accurately describe existing data from variable-ratio schedules. The theory accounts for two additional characteristics of behavior maintained by variable-ratio schedules; namely, the appearance of strained, two-valued (i.e., zero or very rapid) responding at large $\bar{n}$s, and the abrupt cessation of responding at a boundary $\bar{n}$. The theory also accounts for differences between behavior on variable-interval and variable-ratio schedules, including (a) the occurrence of strained responding on variable-ratio but not on variable-interval schedules, (b) the abrupt cessation of responding on variable-ratio but not on variable-interval schedules even at extreme parameter values, and (c) the occurrence of higher response rates on variable-ratio than on variable-interval schedules. Furthermore, given data from a series of variable-interval schedules and from a series of concurrent variable-ratio variable-interval schedules, the theory permits quantitative prediction of many properties of behavior on single-alternative variable-ratio schedules. The linear system theory's combined account of behavior on variable-interval and variable-ratio schedules is superior to existing versions of six other mathematical theories of variable-interval and variable-ratio responding.

Key words: variable-ratio schedules, linear system theory, feedback, ratio strain, bias

McDowell and Kessel (1979) used the mathematical theory of linear systems (Aseltine, 1958) to describe behavior maintained by variable-interval (VI) schedules of reinforcement. The linear system theory is a set of mathematical techniques that can be used to calculate the response of a system to a known input, provided the system can be described at least in principle by a linear differential equation. The first step in applying the theory is to write the reinforcement input delivered to the organism and the response output produced by the organism in explicit mathematical forms. McDowell and Kessel (1979) wrote the reinforcement input for the VI case as a train of rectangular pulses spaced irregularly in time. Figure 1 is a plot of a section of the function, designated $R(t)$, or reinforcement as a function of time. Each rectangular pulse represents a single reinforcement, and is described by an amplitude, A_R, and a width, w. The ordinate of $R(t)$ is a valuelike dimension. Reinforcers with higher values may be represented by pulses with higher amplitudes. Figure 1 shows that $R(t) = A_R$ during each reinforcement pulse and that $R(t) = 0$ at all other times. The transitions between the two values are jump discontinuities. McDowell and Kessel wrote the response output on a VI schedule in the same way. The output function, designated $B(t)$, or behavior as a function of time, consists of a train of rectangular response pulses spaced irregularly in time. Each response pulse is characterized by an amplitude, A_B, and a width, w^*. The ordinate of $B(t)$ is a valuelike dimension such that responses of greater aversiveness may be represented by pulses with higher amplitudes. Like $R(t)$, $B(t)$ is a two-valued function with discontinuous transitions between A_B and zero. Both $R(t)$ and $B(t)$ for the VI case have the additional property that the mean time between reinforcement or response pulses is constant when large numbers of pulses are con-

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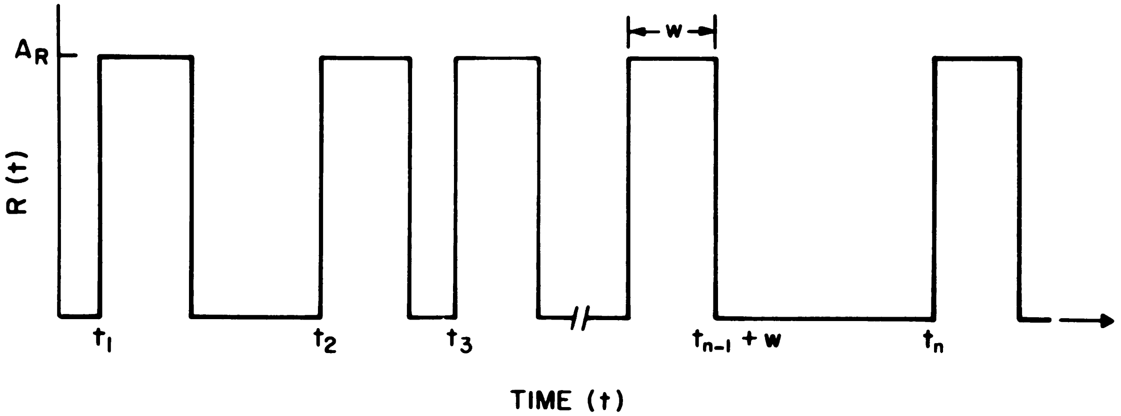

Fig. 1. A train of reinforcement pulses. $R(t)$ is a valuelike dimension. Pulses start at irregularly spaced times, t_1, t_2, t_3, t_{n-1}, and t_n, where the index, $n = \{1, 2, 3, \ldots\}$. The amplitude of each pulse is A_R. The duration of each pulse is w. (From "Variable-interval rate equations and reinforcement and response distributions" by J. J McDowell, R. Bass, and R. Kessel, 1983, *Psychological Review*, **90**, 364–375. Copyright 1983 by the American Psychological Association, Inc. Reprinted by permission.)

sidered. This is a well-known characteristic of VI reinforcement and responding (McDowell & Sulzen, 1981).

The linear system analysis follows routinely from the input and the output functions, $R(t)$ and $B(t)$. The calculations are usually made more tractable by applying a function transform known as the Laplace transform. Function transforms are familiar to many researchers in the context of proportional ratio matching. The power-function version of the matching equation is usually subjected to logarithmic transformation (de Villiers, 1977), which converts the equation into a simpler, linear form. Logarithmic transforms simplify functions by changing multiplication and division into addition and subtraction. Laplace transforms simplify functions by changing integration and differentiation into multiplication and division.

Given that a system can be described by a linear differential equation, it can be shown that the ratio of the Laplace transform of the output of the system to the Laplace transform of the input to the system is constant (Aseltine, 1958). Thus, for $R(t)$ and $B(t)$,

$$\frac{\mathscr{L}[B(t)]}{\mathscr{L}[R(t)]} = \gamma, \tag{1}$$

where $\mathscr{L}[f]$ represents the Laplace transform of the indicated function. The scalar constant, γ, represents properties of the organismic system that are constant with respect to time. McDowell and Kessel (1979) obtained a mean-value rate equation (cf. McDowell, Bass, & Kessel, 1983) for the VI case by calculating the Laplace transforms of $B(t)$ and $R(t)$, substituting them into Equation 1, and solving for the average rate of responding, R_{out}. The result was

$$R_{\text{out}} = \left\{ \ln\left[1 + \frac{P_B}{\gamma P_R}\left(e^{1/R_{\text{in}}+w} - 1\right) \right] - w^{*} \right\}^{-1}, \tag{2}$$

where R_{in} represents the average rate of reinforcement, P_B represents the aversiveness of the response, and P_R represents the value of the reinforcer. The quantity P_B is defined as the definite integral of $B(t)$ during one response pulse, and the quantity P_R is defined as the definite integral of $R(t)$ during one reinforcement pulse (McDowell, 1987). These integrals incorporate the amplitude parameters A_B and A_R. According to Equation 2, the rate of responding on VI schedules varies not only with the rate of reinforcement, but also with the value of the reinforcer and with the aversiveness of the response.

The description of VI responding provided by Equation 2 has proved to be excellent (McDowell, 1980, 1987; McDowell & Kessel, 1979) and superior to the descriptions provided by seven other mathematical accounts of the VI case (Catania, 1973; Killeen, 1981, 1982; Rachlin, 1978; Staddon, 1977, 1979), including Herrnstein's (1970) matching-based account (McDowell et al., 1983; McDowell & Kessel, 1979; McDowell & Wood, 1984, 1985). Although several properties of Equation 2 remain to be tested (McDowell, 1987), the success of the linear-system description of the VI case indicates that applications of the theory to other cases might also be successful. In the present article we apply the linear system theory to the case of behavior maintained by variable-ratio (VR) schedules.

THE VI-PLUS-LINEAR-FEEDBACK APPROACH TO THE VR CASE

McDowell (1979, 1980) pointed out that the simplest mathematical approach to the VR case was to treat it as VI responding, but with the addition of a feedback loop. Indeed, the functions $R(t)$ and $B(t)$ for the VI case also describe the reinforcement input and response output for the VR case. On both types of schedule, reinforcements and responses are spaced irregularly in time and the mean time between reinforcement or response events is roughly constant. Missing from the VI analysis, however, is the feedback loop that links reinforcement rate to response rate on VR schedules. Of course feedback also occurs on VI schedules, but it is restricted to a small range of low response rates and is minimal in comparison to VR feedback, which operates at all response rates (cf. McDowell, 1980).

The VR feedback loop is determined by the average ratio requirement, $\bar{n}$, that the VR schedule arranges. If this average requirement is emitted slowly reinforcers will be delivered slowly, and if it is emitted rapidly reinforcers will be delivered rapidly. The direct relationship between reinforcement rate and response rate on VR schedules is described by the feedback function,

$$R_{\text{in}} = (1/\bar{n})R_{\text{out}}, \tag{3}$$

which is a line with intercept equal to zero and slope equal to the reciprocal of the average ratio requirement. For VR schedules, Equation 3 is true by definition because it describes a defining, or necessary, property of these schedules. However, the linear feedback described by Equation 3 can be arranged independently of a ratio contingency because $1/\bar{n}$ is simply a number that can assume any finite value greater than zero.

Because Equation 2 was written to describe responding on simple VI schedules, it is evident that the linear system theory requires the composite of Equation 3 on Equation 2 to describe responding on a VI schedule to which a linear feedback loop has been added. This type of schedule is not difficult to arrange. Reinforcement on a VI-plus-linear-feedback schedule is delivered according to a time-based, or interval, contingency, as on a simple VI schedule. However, reinforcement rate on this type of schedule increases linearly with response rate according to Equation 3. In other words, throughout the session the effective average interreinforcement interval (or VI value, i.e., $1/R_{\text{in}}$) varies directly with the average interresponse time ($1/R_{\text{out}}$). Higher response rates produce smaller mean VI values (i.e., higher reinforcement rates) and lower response rates produce larger mean VI values (i.e., lower reinforcement rates). According to the linear system theory, responding on this type of schedule must occur in the following manner. An initial response rate produces an initial reinforcement rate according to Equation 3. This reinforcement rate produces a new response rate according to Equation 2, which in turn produces a new reinforcement rate according to Equation 3, and so on. The equilibrium condition is obtained by substituting Equation 3 into Equation 2 and solving for R_{out}. The result, which will be discussed in detail later, is an equation that expresses equilibrium response rate as a function of $\bar{n}$.

In McDowell's (1979, 1980) suggested approach to the VR case, VR schedules are treated mathematically as if they were VI schedules with linear feedback loops. In other words, the two types of schedule are assumed to be equivalent. If this assumption is valid, response outputs produced by VI-plus-linear-feedback schedules should be indistinguishable from those produced by ordinary VR schedules. For example, the high response rates that VR schedules are known to generate (Baum, 1981) should also be produced by VI schedules with linear feedback loops. As another example, some researchers (e.g., Green, Kagel, & Battalio, 1982; but cf. Mazur, 1983) have reported that the function relating response rate and average ratio requirement on VR schedules is bitonic; that is, response rate first increases and then decreases as the average ratio requirement increases. If this is a reliable property of

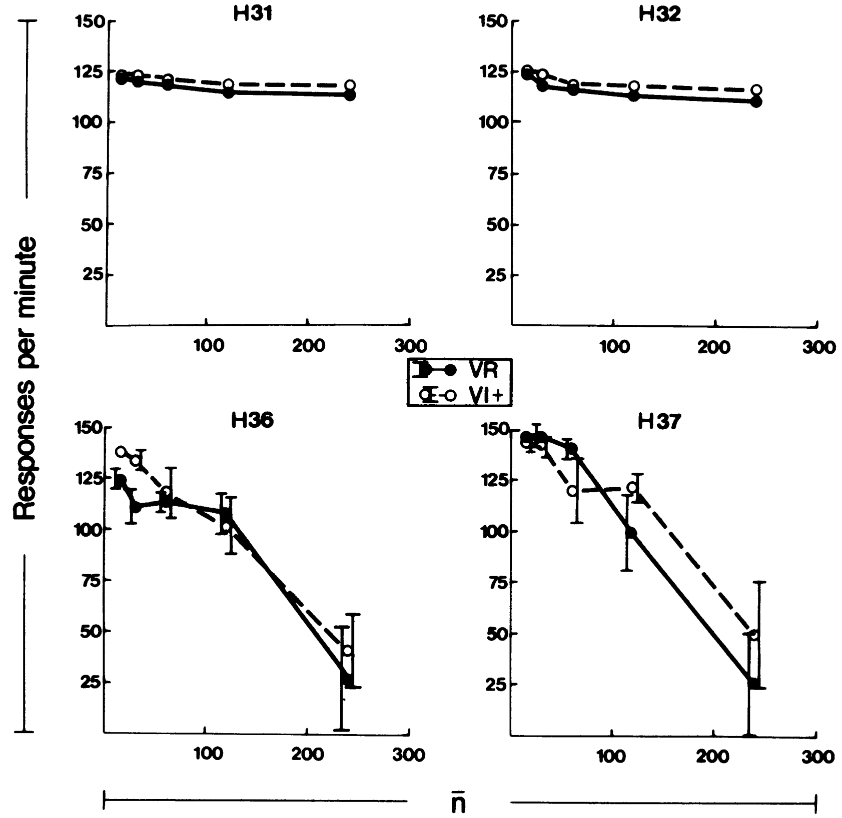

Fig. 2. Average response rates of McDowell and Wixted's (1986) human subjects on VR (filled circles) and VI-plus-linear-feedback (unfilled circles) schedules. Error bars (±1 *SE*) are shown unless they were less than or equal to the diameter of the data point. The quantity, $\bar{n}$, is the reciprocal of the slope of the linear feedback function (Equation 3). For the VR schedules, $\bar{n}$ also represents the mean ratio requirement. (From "Variable-ratio schedules as variable-interval schedules with linear feedback loops" by J. J McDowell and J. T. Wixted, 1986, *Journal of the Experimental Analysis of Behavior,* **46,** 315–329. Copyright by the Society for the Experimental Analysis of Behavior, Inc. Reprinted by permission.)

VR responding, then it should also be observed on VI schedules with linear feedback loops.

The assumption that VR and VI-plus-linear-feedback schedules are equivalent was tested by McDowell and Wixted (1986). In one phase of their experiment, each of 4 human subjects pressed a lever for monetary reinforcement on five VR schedules with mean ratio requirements ($\bar{n}$) of 15, 30, 60, 120, and 240. In a second phase of the experiment, the subjects worked on five VI schedules with linear feedback loops (Equation 3) that reproduced the feedback properties of the VR schedules. It is important to recognize that although the two types of schedule had identical feedback properties, they arranged different types of reinforcement contingencies. The interval contingencies arranged by the VI-plus-linear-feedback schedules permitted subjects to wait until a scheduled interval lapsed before responding. This was not possible on the VR schedules because of the response-based ratio contingencies that these schedules arranged. McDowell and Wixted found that the response outputs produced by the two types of schedule were indistinguishable. Some of the results are shown in Figure 2, where each subject's average response rates on the VR (filled circles) and the VI-plus-linear-feedback schedules (VI+; unfilled circles) are plotted against $\bar{n}$. For each subject the average response rates at a given $\bar{n}$ on the two types of schedule were very similar. The response rate versus $\bar{n}$ functions for the two types of schedule were also very similar. McDowell and Wixted further noted that the similarity in response outputs on the two types of schedule extended to the details of responding. For example, both

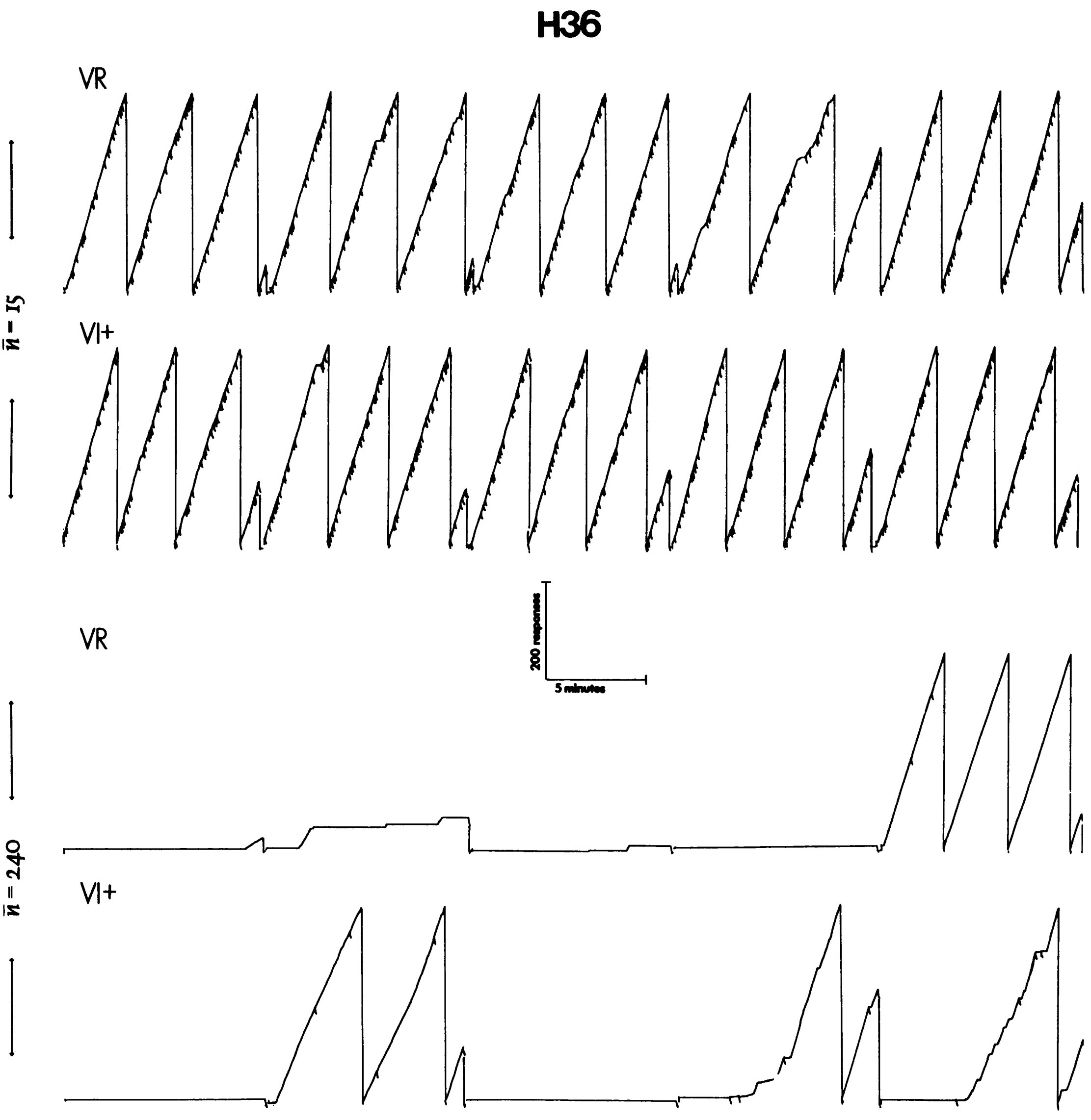

Fig. 3. Cumulative records from one of McDowell and Wixted's (1986) human subjects. The top two records show the subject's final 50 min of responding at $\bar{n}$ = 15 on the VR and the VI-plus-linear-feedback (VI+) schedules. The bottom two records show the subject's final 50 min of responding at $\bar{n}$ = 240 on the two types of schedule. In all records, the pen reset every 10 min, and within 10-min periods it reset every 400 responses. Downward deflections of the pen indicate reinforcer deliveries. (From "Variable-ratio schedules as variable-interval schedules with linear feedback loops" by J. J McDowell and J. T. Wixted, 1986, *Journal of the Experimental Analysis of Behavior,* **46,** 315–329. Copyright by the Society for the Experimental Analysis of Behavior, Inc. Reprinted by permission.)

types of schedule produced high, steady response rates at most $\bar{n}$s, occasional brief pauses that ended in abrupt transitions to the response rate that prevailed before the pause, and, in 2 subjects (H36 and H37), "strained" responding at $\bar{n}$ = 240 that consisted of alternating periods of zero and very rapid response rates (cf. Ferster & Skinner, 1957). All of these details can be seen in Figure 3, which reproduces cumulative records from one of McDowell and Wixted's subjects. The records shown represent the subject's final 50 min of responding at $\bar{n}$ = 15 and at $\bar{n}$ = 240 on the two types of schedule (VI+ = VI-plus-linear-feedback schedule). As illustrated in Figure 3, the cumulative records from the two types of schedule in this experiment were very similar. McDowell and Wixted concluded that VR and VI-plus-linear-feedback schedules are equivalent and, consequently, that the assumption entailed by the VI-plus-linear-feedback approach to the VR case is valid.

Given that this initial assumption has been shown to hold, the next step in pursuing the VI-plus-linear-feedback account of the VR case is to identify the characteristics of VR responding that must be accounted for by the mathematical theory. Following this, the theory will be developed in detail.

THE EMPIRICAL CHARACTERISTICS OF BEHAVIOR MAINTAINED BY VR SCHEDULES

The first important characteristic of behavior on VR schedules is the form of the function relating response rate and $\bar{n}$. Although VR schedules have been used in a variety of contexts in many experiments, only four studies (Brandauer, 1958; Green et al., 1982; Mazur, 1983; McDowell & Wixted, 1986) have yielded enough parametric data from single-alternative VR schedules to permit an examination of the form of the response rate versus $\bar{n}$ function. Before discussing these studies it is necessary to consider what Baum (1981) has referred to as the obligatory postreinforcement pause. This pause consists of three (possibly overlapping) components, namely, the time required to move from the feeder to the operandum, the time after the end of the reinforcement cycle that may be required to finish ingesting the reinforcer, and the time that may be occupied by species-specific postprandial behavior like snout washing. According to Baum (1981), the cumulative obligatory pause time should be subtracted from the session time when calculating response rate. Failure to do so has little effect on the calculated response rate when reinforcers are delivered relatively infrequently, as on VR schedules with moderate to high mean ratio requirements and on most VI schedules. However, when reinforcers are delivered at very high rates, as on VR schedules with low mean ratio requirements, failure to subtract the cumulative obligatory pause time from the session time results in calculated response rates that are spuriously low. To properly evaluate the form of the response rate versus $\bar{n}$ function, Baum recommended that response rates be corrected for obligatory pausing by using the shortest mean postreinforcement pause on a given VR series as an estimate of the obligatory pause. We follow Baum's recommendation here.

The first set of parametric VR data is from McDowell and Wixted's (1986) experiment with human subjects, which was described earlier. The response rates from this experiment do not require correction for obligatory pausing because the nonconsummatory reinforcer used in the experiment did not require movement away from the operandum and could not have entailed ingestion time or time for postprandial behavior. The response rates on the VR and the VI-plus-linear-feedback schedules in this experiment are listed in Table A1 in Appendix A and, as noted earlier, are plotted against $\bar{n}$ in Figure 2. Each subject's response rate declined as $\bar{n}$ increased, although the decline was more marked for H36 and H37 than for the other 2 subjects.

The second set of parametric VR data is from Brandauer's (1958) study of key pecking in pigeons. Three pigeons' responding was reinforced by brief periods of access to mixed grain on VR schedules with mean ratio requirements of 10 to 600 responses per reinforcement. The smallest mean postreinforcement pauses for the 3 birds in numerical order were 0.81, 0.98, and 0.91 s. These mean pauses, each of which occurred at $\bar{n} = 10$, were used to correct the individual birds' response rates. The corrected rates are listed in Table A2 in Appendix A. Bird P17 did not show sustained responding at $\bar{n} = 400$, and neither P14 nor P17 showed sustained responding at $\bar{n} = 600$. In addition, Braundauer discarded a fourth bird (P16) from the experiment because of unstable responding. Also omitted from Table A2 are the response rates on schedules of continuous reinforcement, which are fixed-ratio (FR) rather than VR schedules. The corrected response rates of Brandauer's pigeons are plotted against $\bar{n}$ in the top left panel of Figure 4. Evidently, response rate declined with $\bar{n}$ for the individual birds. There was some indication of bitonicity in P15's function (viz., the increasing limb from the first to the second $\bar{n}$), but this was not confirmed by any other bird's results. The mean corrected response rate across the 3 birds, plotted in the top right panel of Figure 4, shows a smooth decline with $\bar{n}$.

The third set of parametric VR data is from

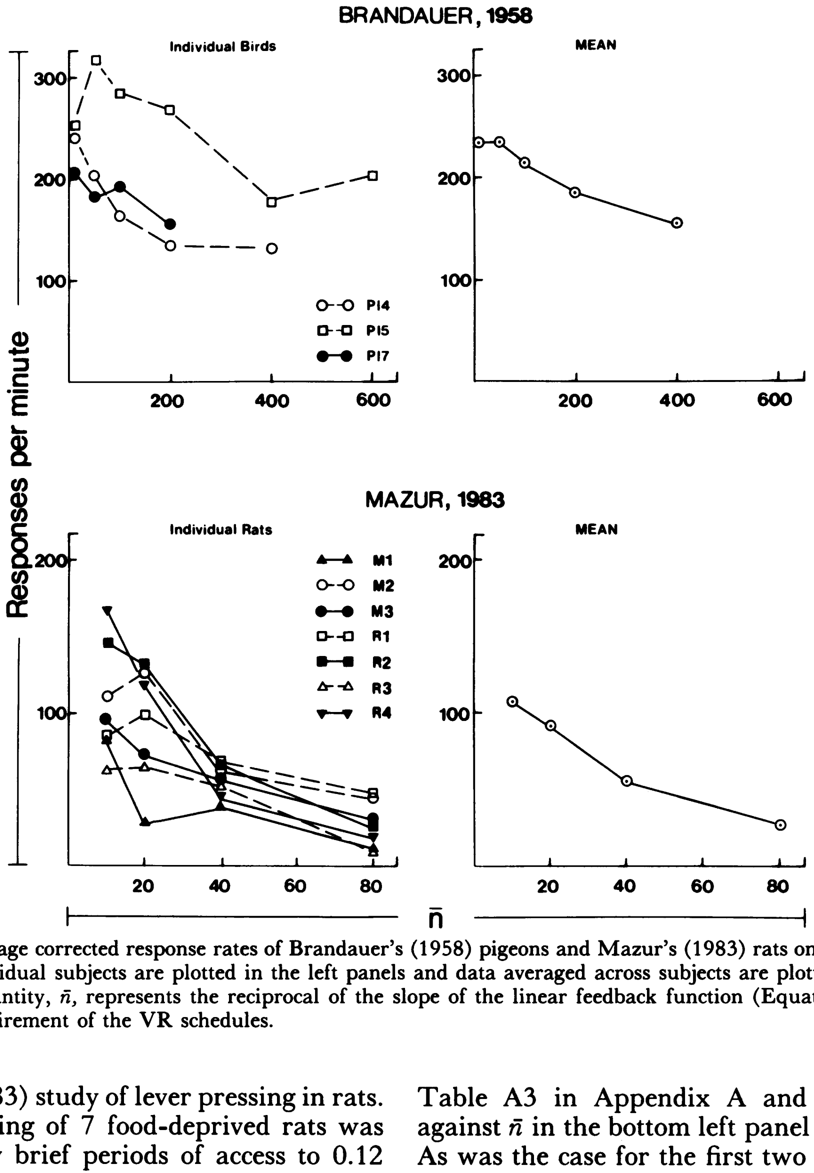

Fig. 4. Average corrected response rates of Brandauer's (1958) pigeons and Mazur's (1983) rats on VR schedules. Data from individual subjects are plotted in the left panels and data averaged across subjects are plotted in the right panels. The quantity, $\bar{n}$, represents the reciprocal of the slope of the linear feedback function (Equation 3) and the mean ratio requirement of the VR schedules.

Mazur's (1983) study of lever pressing in rats. The responding of 7 food-deprived rats was reinforced by brief periods of access to 0.12 mL of sweetened milk on VR schedules with mean ratio requirements of 10 to 80 responses per reinforcement. The smallest mean postreinforcement pauses for M1 through M3 were 2.13, 1.71, and 3.33 s. These mean pauses, which were obtained at $\bar{n}$ = 10, 40, and 20, respectively, were used to correct each rat's individual response rates. The smallest mean postreinforcement pauses for R1 through R4 were 2.06, 1.96, 3.08, and 3.75 s. These mean pauses, which were obtained at $\bar{n}$ = 20, 20, 10, and 10, respectively, were used to correct each of these rats' individual response rates. The corrected rates for all rats are listed in Table A3 in Appendix A and are plotted against $\bar{n}$ in the bottom left panel of Figure 4. As was the case for the first two sets of data, response rate declined with $\bar{n}$ for each subject in this experiment. The individual functions for M2 and R1 showed slight bitonic trends. In addition, the rate of decline of response rate with $\bar{n}$ varied considerably among subjects (as in McDowell & Wixted's, 1986, experiment). For example, R3's rate of decline was modest, whereas R4's was marked. The mean corrected response rate across the 7 rats, plotted in the lower right panel of Figure 4, shows a clear monotonic decline with $\bar{n}$.

The final set of parametric VR data is from Green et al.'s (1982) study of key pecking in pigeons. The responding of 4 pigeons was reinforced by brief periods of access to mixed grain on VR schedules with mean ratio requirements of 12.5 to 400 responses per reinforcement. Because Green et al. did not report mean postreinforcement pauses, the median of the smallest mean postreinforcement pauses from Brandauer's (1958) experiment (viz., 0.91 s) was used to correct the response rates of Green et al.'s pigeons. The corrected rates are listed in Table A4 in Appendix A. Inspection of these data shows that the response rate versus $\bar{n}$ functions were markedly bitonic for 3 of the 4 pigeons. Except for P47, the birds' response rates increased from $\bar{n} = 12.5$ to $\bar{n} = 50$ and then decreased as $\bar{n}$ increased further. Because of the constant 40-min session lengths in Green et al.'s experiment, it is possible that these bitonic functions were artifacts of satiation at the two lowest $\bar{n}$s. For example, Brandauer (1958) found that at low $\bar{n}$s sessions as brief as 15 min produced so many reinforcers that his birds gained weight rapidly. To prevent satiation, he ended sessions after a specified number of reinforcements—usually 50, but never more than 100. The reinforcements per session in Green et al.'s experiment are listed in Table A4 in Appendix A. Approximately 200 reinforcers were delivered in each session at $\bar{n} = 25$ and approximately 300 reinforcers were delivered in each session at $\bar{n} = 12.5$. These numbers far exceeded the maximum that Brandauer found advisable. Evidently, satiation cannot be ruled out as an explanation of the bitonic function forms in Green et al.'s experiment.

To summarize, data from the humans in McDowell and Wixted's (1986) experiment and from the pigeons and rats in Brandauer's (1958) and Mazur's (1983) experiments show that response rate on VR schedules is a decreasing function of the mean ratio requirement, $\bar{n}$. Green et al.'s (1982) contradictory results may be an artifact of satiation.

In addition to the decreasing form of the response rate versus $\bar{n}$ function, two further characteristics of VR responding have been reported with some frequency. One is the occurrence of strained, two-valued (i.e., nearly zero or very rapid) responding at high mean ratio requirements (Baum, 1981). This phenomenon was first reported by Ferster and Skinner (1957) in 2 pigeons responding on VR schedules at $\bar{n} = 360$, and in a third pigeon responding on a VR schedule with a mean ratio requirement of roughly 380 (estimated from their Figure 487A). Similarly, on the basis of his postreinforcement pause data, Braundauer (1958) argued that responding becomes unstable on VR schedules at high $\bar{n}$s. Finally, 2 of the 4 human subjects in McDowell and Wixted's (1986) experiment showed strained responding on VR schedules at $\bar{n} = 240$.

The second additional characteristic of VR responding is the existence of a boundary $\bar{n}$ beyond which responding fails to occur. Ferster and Skinner (1957), Baum (1981), and Zeiler (1977, 1979) have discussed this phenomenon. As one example, recall that in Brandauer's (1958) experiment, responding could not be maintained in 1 bird at $\bar{n} = 400$ and could not be maintained in another at $\bar{n} = 600$. Similarly, based on the cumulative records in McDowell and Wixted's (1986) experiment, it seems unlikely that H36's and H37's responding could have been maintained at $\bar{n}$s much larger than 240.

Besides incorporating the three characteristics of VR responding just discussed, a satisfactory mathematical theory must account for the differences between behavior on VR schedules and behavior on VI schedules. One difference is that strained responding occurs on VR schedules but not on VI schedules, even when the mean interreinforcement interval of the VI schedule is large (Baum, 1981). A second difference is that responding ceases abruptly at a boundary $\bar{n}$ on VR schedules but does not cease abruptly under any circumstance on VI schedules (Baum, 1981; Zeiler, 1977, 1979). A third difference is that responding occurs at a higher rate on VR than on VI schedules (Catania, Matthews, Silverman, & Yohalem, 1977; Ferster & Skinner, 1957; Matthews, Shimoff, Catania, & Sagvolden, 1977; Mazur, 1983; McDowell & Wixted, 1986; Zuriff, 1970).

THE MATHEMATICAL THEORY OF BEHAVIOR MAINTAINED BY VR SCHEDULES

In this section the VI-plus-linear-feedback account of the VR case will be developed formally; it will then be evaluated against the empirical characteristics of VR responding that were identified in the last section. Following this individual treatment of the VR case, the formal and empirical differences between behavior on VR and VI schedules will be discussed.

VR Schedules

As noted earlier, the VI-plus-linear-feedback approach to the VR case is reasonable from a purely formal point of view. The functions $R(t)$ and $B(t)$ for the VI case also describe reinforcement and responding on VR schedules, and the addition of a linear feedback loop incorporates the feedback that VR schedules necessarily arrange. According to this view, the composite of Equation 3, the linear feedback function, on Equation 2, the mean-value rate equation (which is obtained from $R(t)$ and $B(t)$ for the VI case), should provide a complete description of behavior on VR schedules.

To develop this account, it will be convenient to rewrite Equation 2 as

$$R_{\text{out}} = \left\{ \ln \left[m e^{1/R_{\text{in}}} + b \right] \right\}^{-1}, \qquad (4)$$

where $m = (P_B e^{-w^*})/(\gamma P_R e^{-w})$ and $b = e^{-w^*} - (P_B e^{-w^*})/(\gamma P_R)$. Equation 4 is just an algebraic rearrangement of Equation 2; its complete derivation is given in Appendix B. Equation 4 is written to emphasize the relationship between response rate, R_{out}, and reinforcement rate, R_{in}. This relationship, in the form specified by Equation 4, is known to hold for VI schedules (McDowell, 1980; McDowell & Kessel, 1979). According to the present account it must also hold for VR schedules. That is, a particular reinforcement rate on a VR schedule must produce the response rate required by Equation 4. Of course, on VR schedules reinforcement rate is not under the experimenter's control but is generated by the organism according to Equation 3, the linear feedback function. The slope of this function (or its reciprocal, $\bar{n}$) is the true independent variable. We may obtain an expression relating response rate and $\bar{n}$ by substituting Equation 3 into Equation 4:

$$R_{\text{out}} = \left\{ \ln \left[m e^{\bar{n}/R_{\text{out}}} + b \right] \right\}^{-1}. \qquad (5)$$

This equation cannot be solved analytically for R_{out}. However, given specific values of $\bar{n}$ and of the parameters m and b, it can be solved by numerical techniques such as Newton's method (e.g., Heading, 1970).

According to the present account of the VR case, Equation 5 must describe the relationship between equilibrium response rate and $\bar{n}$ on VR schedules, and Equation 4 must describe the relationship between equilibrium response rate and reinforcement rate on these schedules. The latter must be the case even though reinforcement rate on VR schedules is not under experimental control. It should also be clear that, regardless of the appropriateness of the VR analysis, the linear system theory requires Equations 4 and 5 to describe responding on VI schedules to which linear feedback loops have been added.

It will be helpful to first examine the general forms of Equations 4 and 5, examples of which are shown in Figure 6. The curves in the right panels of the figure are plots of Equation 4, the general form of which has been discussed in detail elsewhere (McDowell, 1979, 1980, 1987; McDowell & Kessel, 1979). The curves in the left panels are plots of Equation 5. It is evident from these plots that Equation 5 entails a boundary $\bar{n}$, beyond which the equation has no finite positive solution. For positive values of $\bar{n}$ less than the boundary $\bar{n}$, two finite response rates greater than zero satisfy the equation. One of these rates is small (near zero), the other is much larger, and the two approach each other, the larger rate changing more rapidly, as $\bar{n}$ increases toward its boundary value.

The source of the unusual back-bending form of Equation 5 is a point of inflection in Equation 4 that occurs at a very low reinforcement rate (McDowell, 1979). At the rate equation's point of inflection its curvature changes from concave upward to concave downward, its first derivative (or slope) is a maximum, and its second derivative is zero. The rate equation's point of inflection and relevant differential properties are discussed in more detail in Appendix B.

The top panel of Figure 5 illustrates how the rate equation's point of inflection interacts with the linear feedback function, Equation 3, to produce forms like those shown in the left panels of Figure 6. The curve in the top panel of Figure 5 is a stylized plot of Equation 4 that greatly exaggerates its change in curvature. In an actual plot of Equation 4 (as shown, e.g., in the right panels of Figure 6), the point of inflection occurs at such a small reinforcement rate that the change in curvature is not visually detectable for typical ranges of rein-

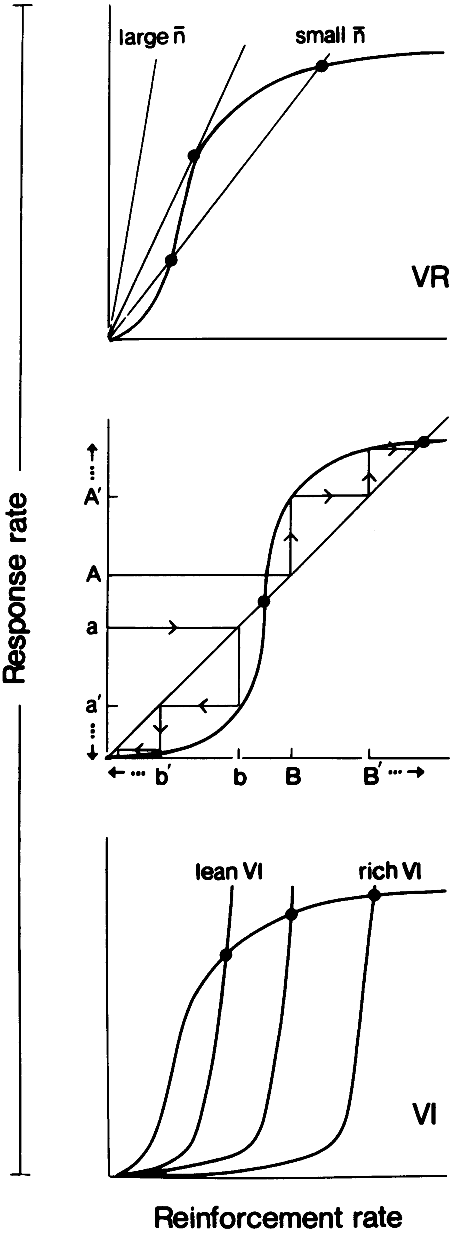

Fig. 5. The rate equation and feedback functions for VR (top and middle panels) and VI (bottom) schedules. In all panels the curve with the response-rate asymptote is a stylized representation of the rate equation. The equation's point of inflection has been moved far to the right and its change in curvature has been greatly exaggerated. In all panels, filled circles represent equilibrium, or steady-state, response rates. The lines in the top panel are VR feedback functions; the curves with reinforcement-rate asymptotes in the bottom panel are stylized representations of VI feedback functions. The operation of a VR feedback loop is illustrated in the middle panel. The loop drives an initial response rate, *A*, to the next higher equilibrium rate (filled circle), and it drives an initial response rate, *a*, to zero. Notice that the feedback loop always drives responding away from the lower nonzero equilibrium rate.

forcement and response rates. Accompanying the stylized plot of the rate equation in the top panel of Figure 5 are three examples of a linear feedback function, Equation 3. In Figure 5's coordinates the feedback function becomes more nearly vertical as $\bar{n}$ increases. At points where the rate equation and a feedback function intersect (filled circles), Equations 4 and 3 are satisfied simultaneously, so Equation 5 is also satisfied. According to the present theory, response rates at these points may be observed in the steady state. Response rates not represented by points of intersection can be observed only in transition states. As shown in the top panel of Figure 5, two widely separated response rates satisfy Equation 5 at small $\bar{n}$s. As $\bar{n}$ increases and the linear feedback function becomes more nearly vertical, the two response rates that satisfy Equation 5 approach each other. When the feedback function is tangent to the rate equation, as illustrated by the middle line in the top panel of Figure 5, only one equilibrium response rate satisfies Equation 5. As $\bar{n}$ increases still further, the feedback function and the rate equation cease to intersect in the first quadrant, which means that Equations 4 and 3 cannot be satisfied simultaneously by any positive response rate and that, consequently, Equation 5 does not have a finite positive solution. If all the response rates that simultaneously satisfy Equations 4 and 3 (and that as a consequence satisfy Equation 5) are plotted against their respective $\bar{n}$s, the plot takes a form like that shown in the left panels of Figure 6. The boundary $\bar{n}$ in these plots is the reciprocal of the slope of the feedback function that is tangent to the rate equation.

It is important to recognize that the unusual form of Equation 5 is due to Equation 4's change in curvature. If Equation 4 were concave downward for all positive reinforcement rates (as is, e.g., Herrnstein's, 1970, hyperbola), then every linear feedback function would intersect the equation exactly once, and no feedback function could be tangent to it in the first quadrant. Thus, a single equilibrium response rate would be associated with each $\bar{n}$, and as $\bar{n}$ increased, the equilibrium response rate would decrease continuously to zero as opposed to jumping discontinuously to zero at a boundary value.

Consider now those VR schedules for which Equation 5 has two solutions. The dynamics of the linear feedback loop ensure that only one of the two response rates that satisfy Equa-

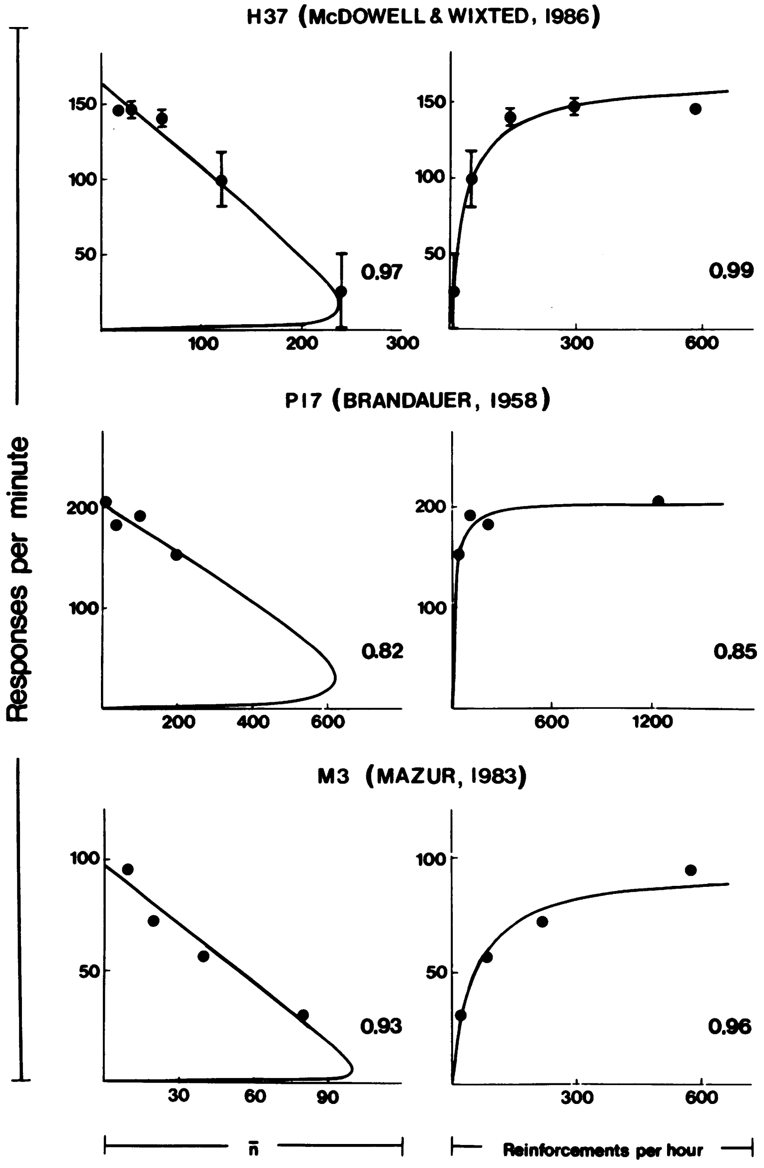

Fig. 6. Response rates of a human (top panels) and corrected response rates of a pigeon (middle panels) and of a rat (bottom panels) on various VR schedules. In the left panels the response rates are plotted against $\bar{n}$; in the right panels they are plotted against the scheduled reinforcement rates, as explained in the text. The error bars in the top panels represent ±1 *SE*. The smooth curves in the left and right panels are plots of Equations 5 and 4, respectively. The proportion of variance accounted for by the fitted equation is given in each panel.

tion 5 is likely to be observed under most circumstances. The operation of this loop is illustrated in the middle panel of Figure 5. The curve is a stylized representation of Equation 4, the shape of which is greatly distorted so that the operation of the linear feedback loop can be illustrated clearly. The straight line is a linear feedback function, and the filled circles represent the two response rates that satisfy Equation 5. The feedback loop operates as follows. An initial response rate, A, produces a reinforcement rate, B, according to the linear feedback function. But this reinforcement rate generates a higher response rate, A', according to the rate equation. The higher response rate produces a higher reinforcement rate, B', which in turn generates an even higher response rate, and so on, until responding reaches the equilibrium rate indicated by the filled circle in the upper right corner of the panel. Similarly, an initial response rate, a, produces a reinforcement rate, b, according to the linear feedback function. But this reinforcement rate only supports a lower response rate, a', according to the rate equation. The lower response rate produces a lower reinforcement rate, b', which in turn supports an even lower response rate, and so on, until responding is driven to zero. In general, a nonequilibrium response rate that occurs in a region where the rate equation lies above the feedback function (e.g., at A) is driven upward to the next higher equilibrium rate. Similarly, a nonequilibrium response rate that occurs in a region where the rate equation lies below the feedback function (e.g., at a) is driven downward to the next lower equilibrium rate, which may be zero.

It follows from the dynamics of the feedback loop that, when responding is at the higher equilibrium rate, any moderate change in rate will be resisted by the loop in such a way that responding will be returned to the equilibrium rate. Consequently, responding at the higher equilibrium rate will tend to be stable. Similarly, when responding is at zero, the development of a moderately positive response rate will be resisted by the feedback loop in such a way that responding will be returned to zero. Hence responding at zero will also tend to be stable. In contrast to these two cases, when responding is at the lower nonzero equilibrium rate, any moderate change in rate will be enhanced by the feedback loop. Responding will be driven away from this equilibrium rate, upward toward the higher equilibrium rate, or downward toward zero, depending on the direction of the initial change in rate. Thus responding at the lower nonzero equilibrium rate is unlikely to be observed in the steady state. Although zero is a stable equilibrium, responding at this rate is also unlikely to be observed in the steady state, as long as $\bar{n}$ is small or moderate. At small and moderate $\bar{n}$s, the two nonzero equilibria are widely separated, and the lower nonzero equilibrium is close to zero. Consequently, most response rates occur in the region where the rate equation lies above the feedback function and hence will be driven to the higher equilibrium rate. For the few (very low) response rates that occur in the region where the rate equation lies below the feedback function, ordinary variability in responding may produce a rate that exceeds the lower nonzero equilibrium rate, in which case responding again would be driven to the higher equilibrium rate.

The foregoing analysis shows that, according to the theory, when two response rates satisfy Equation 5, the higher rate will usually be observed in the steady state. There are two exceptions to this rule. The first occurs when behavior is highly variable. In this circumstance, responding at the higher equilibrium rate may drop temporarily below the lower nonzero equilibrium rate, even when the two rates are widely separated. If this happens, responding will be driven to zero. Alternatively, responding at zero may rise temporarily above the lower nonzero equilibrium rate, in which case it will be driven to the higher equilibrium rate. Thus, when behavior is highly variable, responding may oscillate between the higher equilibrium rate and zero. The second exception to the rule occurs when the three equilibria (including zero) are close together, which happens when $\bar{n}$ is near its boundary value. In this circumstance, ordinary response rate variability may produce the same oscillation between the higher equilibrium rate and zero. As before, this is because responding at the higher rate may drop temporarily below the lower nonzero equilibrium rate, and responding at zero may rise temporarily above it. To describe the details of the time course of this oscillation, it will be necessary to develop a theory of behavioral dynamics. McDowell (1979) has discussed the distinction between dynamic and equilibrium theories.

To summarize, the linear system theory requires response rate to vary with $\bar{n}$ according to Equation 5. When the equation has two solutions, the theory requires the higher response rate to be observed in the steady state under most circumstances. However, as $\bar{n}$ nears its boundary value, responding may oscillate between the higher response rate and zero. A similar oscillation in behavior may be observed at smaller $\bar{n}$s if response rate is highly variable. When $\bar{n}$ exceeds its boundary value, the theory requires responding to drop discontinuously to zero.

Comparison with Data

It is evident that the linear system theory accounts for the occurrence of strained, two-valued responding at high mean ratio requirements and for the existence of a boundary $\bar{n}$ beyond which responding fails to occur. These are the second and third empirical characteristics of VR responding that were identified in the last section. It remains to examine the empirical adequacy of Equations 4 and 5 in describing existing data. Obviously, Equation 5 must describe the declining response rate versus $\bar{n}$ function that was identified earlier.

The empirical adequacy of Equations 4 and 5 was examined by fitting the equations to the parametric data from Brandauer (1958), Green et al. (1982), Mazur (1983), and McDowell and Wixted (1986). Scheduled $\bar{n}$s were used in the fits of Equation 5, and "scheduled" reinforcement rates calculated from Equation 3 with $\bar{n}$ set equal to its scheduled value were used in the fits of Equation 4. The two equations were fitted jointly such that a single pair of parameter values, m and b, was obtained for both equations. Given these parameter values, Equation 4's point of inflection and the boundary $\bar{n}$ entailed by Equation 5 were calculated. The least-squares fitting procedure, and the methods for calculating Equation 4's point of inflection and Equation 5's boundary $\bar{n}$, are discussed in detail in Appendix C. The estimates of m and b obtained for each data set are listed in Table C1 in Appendix C.

The results of the fits of Equations 4 and 5 are summarized in Table 1. The proportions of variance accounted for (pVAF) by the equations, listed in the last two columns of the table, show that Equations 4 and 5 described the data well. Including Green et al.'s (1982) questionable data, Equation 4 accounted for a median of 86% of the individual-subject data variance, and Equation 5 accounted for a median of 81% of the individual-subject data variance. Excluding Green et al.'s data, the median percentages of variance accounted for by Equations 4 and 5 were 92% and 84%, respectively.

In McDowell and Wixted's (1986) experiment with human subjects, Equations 4 and 5 described the data from the VR and the VI-plus-linear-feedback schedules equally well. The proportions of variance accounted for on both types of schedule were smaller for H31 and H32 than for H36 and H37 because there was less variance to account for in the former subjects' data (as is apparent in Figure 2). In Brandauer's (1958) experiment, the individual-subject data varied somewhat about the fitted equations (see Figure 4), but when response rates were averaged across birds, the equations accounted for nearly all of the variance. In Mazur's (1983) experiment the equations described most of the individual-subject data quite well, and accounted for nearly all of the variance when response rates were averaged across rats. In Green et al.'s (1982) experiment Equations 4 and 5 accounted for some of the individual-subject data poorly. Except for P47, the deviations from the fitted equations were systematic, as discussed earlier. Although Equations 4 and 5 accounted for a fair proportion of the variance when the data were averaged across birds, the deviations in this case were also systematic (as can be seen in Table A3 in Appendix A).

The pVAFs in Table 1 show that Equations 4 and 5 typically accounted for similar proportions of the data variance in their respective domains. The large discrepancies in the two pVAFs for R4 in Mazur's (1983) experiment and for P47 in Green et al.'s (1982) experiment were due to the omission of a data point in calculating the pVAFs for Equation 5. In these two cases (as well as for R2 in Mazur's experiment), the largest $\bar{n}$ exceeded the boundary $\bar{n}$ and, consequently, a residual could not be obtained for the response rate at the largest $\bar{n}$. In calculating the pVAFs, this data point was omitted for Equation 5 but not for Equation 4, which means that the former equation had less variance to account for.

Examples of the fits of Equations 4 and 5 are shown in Figure 6. The response rates of a human, a pigeon, and a rat on VR schedules are shown in the top, middle, and bottom panels of the figure. The subjects whose data are plotted are identified in the figure. The smooth curves in the left and right panels are plots of Equations 5 and 4 respectively, and the proportion of variance accounted for by the plotted equation is given in each panel. As the examples in Figure 6 illustrate, Equations 4 and 5 provided a good visual description of the data.

The coordinates of Equation 4's point of inflection and of Equation 5's boundary VR schedule are listed in Table 1 for each data set. Notice that the points of inflection invariably occurred at very low reinforcement and response rates. The boundary $\bar{n}$ represents the largest mean ratio requirement that can support responding for a given subject, according to the theory. Evidently, the response rate at the boundary $\bar{n}$ represents the lowest nonzero

Table 1

Coordinates of Equation 4's point of inflection (rft/hr = reinforcements per hour; rsp/min = responses per minute), coordinates of the boundary VR schedule entailed by Equation 5, and proportions of variance accounted for (pVAF) by Equations 4 and 5, for joint fits of the two equations to data from McDowell and Wixted (1986; VI+ = VI plus linear feedback), Brandauer (1958), Mazur (1983), and Green et al. (1982). All quantities in the table were calculated from unrounded data.

	Point of inflection		Boundary VR		pVAF	
Subject	rft/hr	rsp/min	$\bar{n}$	rsp/min	Eq. 4	Eq. 5
			McDowell and Wixted, 1986			
H31 (VR)	0.6	13.2	1,677	36.0	0.84	0.83
H31 (VI+)	0.5	15.3	2,192	41.0	0.85	0.84
H32 (VR)	0.7	11.6	1,307	32.2	0.79	0.79
H32 (VI+)	0.6	13.7	1,764	37.3	0.76	0.75
H36 (VR)	1.3	4.9	264	14.7	0.94	0.89
H36 (VI+)	1.3	5.2	273	15.7	0.99	0.99
H37 (VR)	1.4	5.0	240	15.3	0.99	0.97
H37 (VI+)	1.3	5.6	300	17.0	0.96	0.93
			Brandauer, 1958			
P14	1.1	9.7	625	30.2	0.78	0.68
P15	1.0	15.4	1,113	47.8	0.73	0.64
P17	1.1	9.6	625	29.7	0.85	0.82
Average data	1.1	11.6	789	36.1	0.97	0.96
			Mazur, 1983			
M1	1.2	1.5	82	3.7	0.77	0.60
M2	1.5	2.4	103	6.8	0.89	0.79
M3	1.4	2.2	100	5.7	0.96	0.93
R1	1.4	2.7	131	7.6	0.87	0.82
R2	1.4	1.6	70	3.8	0.99	0.95[a]
R3	1.2	1.7	87	4.3	0.97	0.95
R4	1.5	1.9	77	4.9	0.95	0.72[a]
Average data	1.4	2.1	91	5.4	0.98	0.94
			Green et al. 1982			
P47	1.2	5.4	317	16.0	0.94	0.72[a]
P48	0.9	9.8	787	29.0	0.48	0.46
P49	0.9	8.8	695	25.8	0.64	0.58
P50	0.8	9.8	903	28.2	0.75	0.73
Average data	1.0	7.9	581	23.4	0.86	0.85
Median of all individual subjects					0.86	0.81

[a] Excluding response rate at the largest $\bar{n}$, which exceeded the boundary $\bar{n}$.

response rate that can be sustained in a given subject on a VR schedule. According to the theory, when $\bar{n}$ is increased beyond the boundary value listed in the table, responding drops discontinuously from the listed rate to zero.

It is clear from the pVAFs in Table 1 and the illustrative plots in Figure 6 that Equation 5 accurately describes the decreasing form of the response rate versus $\bar{n}$ function that characterizes behavior on VR schedules.

Differences Between VR and VI Schedules

The first difference between behavior on VR and VI schedules is that strained responding occurs on the former but not on the latter schedules. The absence of this phenomenon on VI schedules is accounted for by the linear system theory in a straightforward way. Consider the bottom panel of Figure 5, which shows a stylized representation of the rate equation and stylized representations of three VI feedback functions. The exact form of the VI feedback functions is a matter of some dispute. Power functions (Rachlin, 1978), hyperbolas (e.g., Prelec, 1982; Staddon & Motheral, 1978), and other function forms (e.g., Nevin & Baum, 1980) have been proposed. All of the proposed forms have two properties in common, namely, a reinforcement rate asymptote that is approached rapidly, and a first derivative that decreases continuously. These two properties are incorporated in the stylized VI feedback functions plotted in the bottom panel of Figure 5. It is clear from the figure that the points at which the rate equation and the VI feedback functions intersect reproduce the form of the rate equation. Notice that the rate equation always lies above the VI feedback function for response rates less than a given equilibrium rate, and that it always lies below the VI feedback function for response rates greater than a given equilibrium rate. This means that the VI feedback loop always drives changes in response rate back to the equilibrium rate and never to zero or any other rate. It follows from this property of the VI feedback loop that strained responding cannot occur on a VI schedule.

The second difference between behavior on VR and VI schedules is that responding ceases abruptly at a boundary value on the former but not on the latter schedules. This difference is also accounted for by the linear system theory. As illustrated in the bottom panel of Figure 5, every VI feedback function intersects the rate equation exactly once. This means that response rate declines continuously to zero as the VI schedule becomes leaner. In other words, the theory asserts that responding cannot drop discontinuously to zero on a VI schedule, even when the mean interreinforcement interval is very large.

To understand the remaining difference between behavior on VR and VI schedules it is necessary to consider the linear system theory's account of behavior on concurrent schedules. McDowell (1979, 1980, 1987) and McDowell and Kessel (1979) have shown that the theory requires behavior on concurrent schedules to be governed by the following equation:

$$\frac{R_{1\mathrm{OUT}}}{R_{2\mathrm{OUT}}} = \frac{(P_{2B}/P_{2R})}{(P_{1B}/P_{1R})}\left(\frac{R_{1\mathrm{IN}}}{R_{2\mathrm{IN}}}\right). \tag{6}$$

The numerical subscripts refer to the two response alternatives. The uppercase subscripts on the response and reinforcement rates indicate that the widths of the reinforcement and response pulses in $R(t)$ and $B(t)$ have been assumed to be negligible (McDowell, 1979, 1980, 1987; McDowell & Kessel, 1979). Recall that the P_B parameters reflect the aversiveness of responding on the two alternatives and that the P_R parameters reflect the values of the reinforcers. The quotient, P_B/P_R, is a kind of cost-benefit ratio. According to the linear system theory (McDowell, 1987), the value of this ratio reflects all properties of reinforcement and responding that influence behavior on a given alternative, other than reinforcement rate and time-invariant characteristics of the organism (γ). Evidently, Equation 6 is a form of biased matching (Baum, 1974), where bias is given by the quotient of the P_B/P_R ratios. According to Equation 6, bias in favor of the first alternative is observed whenever P_B/P_R is smaller (i.e., whenever the cost-benefit ratio is more favorable) for the first alternative than for the second. Similarly, bias in favor of the second alternative is observed whenever P_B/P_R is smaller for the second alternative than for the first. In practice, the linear system theory treats the concurrent schedule as a method of measuring the P_B/P_R ratios. McDowell (1980, 1987) has explained how numerical values are assigned to these ratios.

Several factors are known to affect bias and hence, according to the linear system theory, the cost-benefit ratios on concurrent schedules. These factors include differences in the type or amount of reinforcement obtained from the component schedules, differences in the type or aversiveness of the response required in each component, and differences in the type of schedule arranged in each component (Baum, 1974). Regarding the last source of bias, Herrnstein and Heyman (1979) found that pigeons' key pecking on concurrent VR VI schedules is biased in favor of the VR schedule. According to the linear system theory, this means that the cost-benefit ratio associated with responding on VR schedules is smaller (i.e., more favorable) than the cost-benefit ratio associated with responding on VI schedules. This may be due to differences in response topography on the two schedules (Baum, 1981), or to differences in the feedback functions that the two schedules entail (McDowell, 1979).

The different VI and VR cost-benefit ratios have important consequences for the linear system theory's account of single-alternative responding. Notice that the cost-benefit ratios in Equation 6 also appear in Equations 2, 4, and 5. Consider Equation 2, which is reproduced here for convenience:

$$R_{\mathrm{out}} = \left\{\ln\left[1 + \frac{P_B}{\gamma P_R}\left(e^{1/R_{\mathrm{in}}+w} - 1\right)\right] - w^*\right\}^{-1}. \tag{2}$$

Recall that Equation 4 is just an algebraic rearrangement of this equation. The linear system theory requires Equation 2 to describe the relationship between response and reinforcement rate on single-alternative VR and VI schedules. But because Herrnstein and Heyman's (1979) results show that the cost-benefit ratio, P_B/P_R, is smaller for VR schedules, Equation 2 requires responding to occur at a higher rate on VR than on VI schedules, all else being equal (i.e., same organism, response, reinforcer, and reinforcement rate). Thus, the linear system theory accounts for the third difference between behavior on VR and VI schedules.

DISCUSSION

We have shown that the linear system theory accounts for the empirical properties of behavior maintained by VR schedules and for the empirical differences between behavior maintained by VR and VI schedules. An important feature of the theory is that it permits quantitative prediction of many properties of responding on single-alternative VR schedules, given data from a series of VI schedules and from a series of concurrent VR VI schedules. The parameters m_{VI} and b_{VI} (subscripted to indicate that they apply to VI schedules) in Equation 4 can be estimated from the VI data. The bias parameter in Equation 6 can be estimated from the concurrent VR VI data. Given known values of m_{VI}, b_{VI}, and of the bias parameter, the values of m_{VR} and b_{VR} (subscripted to indicated that they apply to VR schedules) in Equations 4 and 5 can be calculated. In Appendix D it is shown that

$$m_{\mathrm{VR}} = m_{\mathrm{VI}} \frac{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}$$

and

$$b_{\mathrm{VR}} = 1 + \frac{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}(b_{\mathrm{VI}} - 1),$$

where $(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})/(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})$ is the bias parameter in Equation 6 (assuming the VR schedule is arranged on the second alternative). These values of m_{VR} and b_{VR} completely determine the response rate versus reinforcement rate function (Equation 4) and the response rate versus $\bar{n}$ function (Equation 5) for responding on VR schedules. Thus, all details of single-alternative VR responding can be calculated in advance of obtaining data from these schedules. For example, the response rate at a given $\bar{n}$ can be calculated from Equation 5, and the response rate at a given reinforcement rate can be calculated from Equation 4. The difference between the response rate on a VR and a VI schedule at a given reinforcement rate can be calculated from Equation 4 using the estimated values of m_{VI} and b_{VI} and the calculated values of m_{VR} and b_{VR}. In addition, the boundary $\bar{n}$ beyond which responding will cease, and the response rate at the boundary $\bar{n}$, can be calculated by the method explained in Appendix C.

As noted in the introduction, the linear system theory provides an excellent description of responding on single-alternative VI schedules. Among other phenomena, the theory accounts for the form of the response rate versus reinforcement rate function on VI schedules (McDowell, 1980; McDowell & Kessel, 1979) and for the dependence of the y asymptote of this function on reinforcer value and response aversiveness (McDowell & Wood, 1984, 1985). Thus, the linear system theory provides a unified account of responding on single-alternative VI and VR schedules. There are at least six other mathematical theories of responding on VI and VR schedules. Three of these are based on optimality principles (Baum, 1981; Rachlin, 1978; Staddon, 1979) and three are based on other types of principles, namely, regulation with respect to set points (Hanson & Timberlake, 1983), matching (Herrnstein, 1970; Pear, 1975), and arousal (Killeen, 1982). Each account will be considered briefly here.

Baum's (1981) theory depends on the assumption that organisms maximize net gain, which is a joint function of the reinforcement obtained for responding and the cost associated with it. Cost in Baum's theory increases rapidly with response rate (cf. McDowell, 1979; Rachlin, 1978). Baum showed that his optimality theory describes many features of single-alternative VI and VR responding. However, the predictions of the theory were based on the additional assumption that the y asymptote of the response rate versus reinforcement rate function is constant, an assumption that has been shown to be invalid for VI schedules (McDowell & Wood, 1984, 1985).

Staddon's (1979) theory is based on the assumption that organisms allocate time to instrumental responding such that the distribution of time spent responding, consuming reinforcers, and doing other things is as close as possible to the distribution that would result if the three types of activity were freely available. Although Staddon's theory accounts for a number of properties of VI and VR responding, McDowell and Wood (1984) argued that its description of the form of the response rate versus reinforcement rate function for VI schedules is often inaccurate. In addition, Baum (1981) noted that Staddon's theory does not account for the abrupt cessation of responding on VR schedules at a boundary $\bar{n}$.

Rachlin's (1978) theory is conceptually similar to Staddon's (1979). It is based on the assumption that organisms distribute their time among responding, consuming reinforcers, and doing other things in ways that maximize value, which is a function of the time spent engaging in each of the three types of activity. This theory does not account for the relationship between reinforcer value and the y asymptote of the response rate versus reinforcement rate function (McDowell & Wood, 1984), nor does it unambiguously account for the cessation of responding on VR schedules at a boundary $\bar{n}$ (Baum, 1981).

Among the nonoptimality theories, Herrnstein's (1970) matching theory of VI performance is best known. Pear (1975) extended the theory to VR schedules. Like Baum's (1981) theory, matching theory depends on the assumption that the y asymptote of the response rate versus reinforcement rate function is invariant with respect to changes in reinforcement parameters like magnitude or immediacy. As mentioned earlier, McDowell and Wood (1984, 1985) found that this assumption was violated on VI schedules (but cf. McDowell, 1986). In addition, Baum (1981) argued that matching theory cannot account for the cessation of responding at a boundary $\bar{n}$ (see also Timberlake, 1977).

Killeen's (1982) arousal theory is based on the assumption that reinforcers generate arousal that decays with time but that may cumulate when reinforcers are presented repeatedly. McDowell and Wood (1985) argued that Killeen's theory, in its present form, does not explicitly permit properties of the y asymptote of the response rate versus reinforcement rate function to vary with response aversiveness. Killeen (1982) pointed out problems with his extension of the theory to VR schedules, and he noted that further development of the theory was required.

The final theory was proposed by Hanson and Timberlake (1983). It is based on the assumption that instrumental and contingent responding are regulated with respect to separate set (or base) points, and that under schedule constraint behavior represents a compromise between schedule-produced deviations from the set points. Hanson and Timberlake's theory accounts for certain properties of responding on single-alternative schedules, but apparently it cannot account for the cessation of responding on VR schedules at a boundary $\bar{n}$ or for the higher response rates on VR as compared to VI schedules. In addition, because Hanson and Timberlake do not use specific feedback functions, their theory does not address the form of the response rate versus $\bar{n}$ function on VR schedules.

The linear system theory's account of responding on VI and VR schedules evidently fares well in comparison to the six mathematical accounts reviewed here. Extensions of the theory to other, more complicated, cases are of course possible. Some of these extensions have been discussed by McDowell (1979).

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APPENDIX A

Reinforcement and response rates on VR schedules from experiments by Brandauer (1958), Green et al. (1982), Mazur (1983), and McDowell and Wixted (1986) are listed in the four tables of this appendix. All rates from Brandauer's, Mazur's, and Green et al.'s experiments have been corrected for obligatory pausing (Baum, 1981). As explained in the text, the response rates from McDowell and Wixted's experiment did not require correction for obligatory pausing. Multiple exposures to a single $\bar{n}$ in Green et al.'s and Mazur's experiments were combined by averaging the raw data and then calculating the reinforcement and response rates from the averages. Because Brandauer did not report raw data, multiple exposures in his experiment were combined by averaging the response rates.

Table A1

Mean obtained reinforcement rates (rft/hr = reinforcements per hour) and mean response rates (rsp/min = responses per minute) on VR and VI-plus-linear-feedback schedules from McDowell and Wixted's (1986) 4 human subjects. The quantity, $\bar{n}$, is the reciprocal of the slope of the linear feedback function (Equation 3). For the VR schedules $\bar{n}$ also represents the average ratio requirement. All quantities in the table were calculated from unrounded data.

	VR		VI-plus-linear-feedback	
$\bar{n}$	rft/hr	rsp/min	rft/hr	rsp/min
		H31		
15	493.2	121.9	468.0	123.5
30	255.6	120.1	244.8	123.1
60	135.6	118.6	126.0	121.3
120	56.4	114.8	56.4	119.0
240	34.8	113.9	30.0	118.3
		H32		
15	495.6	124.0	480.0	124.5
30	230.4	118.2	238.8	123.3
60	111.6	116.8	111.6	119.0
120	50.4	113.8	57.6	118.3
240	24.0	110.8	30.0	116.5
		H36		
15	496.8	124.7	530.4	138.4
30	223.2	111.2	267.6	134.0
60	116.4	113.6	122.4	118.3
120	54.0	108.6	56.4	102.3
240	2.4	27.2	14.4	41.0
		H37		
15	584.4	146.0	554.4	143.1
30	289.2	146.4	278.4	141.6
60	145.2	140.4	124.8	120.1
120	52.8	99.7	60.0	121.6
240	7.2	25.4	7.2	49.8

Table A2

Corrected reinforcement and response rates on VR schedules from Braundauer's (1958) 3 pigeons. Each bird's response rates (rsp/min = responses per minute) were corrected for obligatory pausing using the bird's smallest mean postreinforcement pause. These pauses were 0.81, 0.98, and 0.91 s for the 3 birds in numerical order. Because Brandauer did not report obtained reinforcement rates, the reinforcement rates (rft/hr = reinforcements per hour) listed in the table were calculated from Equation 3 using the corrected response rates. Mean corrected reinforcement and response rates across the 3 birds are also listed in the table. The quantity, $\bar{n}$, is the reciprocal of the slope of the linear feedback function (Equation 3) and the mean ratio requirement of the VR schedule. All quantities in the table were calculated from unrounded data.

$\bar{n}$	rft/hr	rsp/min
	P14	
10	1,440.5	240.1
50	242.9	202.4
100	98.4	164.1
200	39.9	133.1
400	19.8	132.1
	P15	
10	1,518.2	253.0
50	379.6	316.3
100	170.3	283.8
200	80.2	267.5
400	26.6	177.3
600	20.1	201.4
	P17	
10	1,235.4	205.9
50	219.2	182.6
100	115.6	192.6
200	46.3	154.5
	Mean	
10	1,398.0	233.0
50	280.5	233.8
100	128.1	213.5
200	55.5	185.0
400	23.2	154.7

Table A3

Corrected reinforcement and response rates on VR schedules from Mazur's (1983) 7 rats. Each rat's response rates (rsp/min = responses per minute) and obtained reinforcement rates (rft/hr = reinforcements per hour) were corrected for obligatory pausing using the rat's smallest mean postreinforcement pause. The pauses were 2.13, 1.71, and 3.33 s for M1 through M3, and 2.06, 1.96, 3.08, and 3.75 s for R1 through R4. Mean corrected reinforcement and response rates across the 7 rats are also listed in the table. The quantity, $\bar{n}$, represents the reciprocal of the slope of the linear feedback function (Equation 3) and the mean ratio requirement of the VR schedules. All quantities in the table were calculated from unrounded data.

$\bar{n}$	rft/hr	rsp/min
	M1	
10	489.8	81.6
20	83.5	28.6
40	57.2	39.0
80	8.5	11.4
	M2	
10	665.0	110.8
20	380.8	126.7
40	91.6	62.7
80	32.2	44.0
	M3	
10	573.7	95.6
20	217.7	72.6
40	83.8	56.5
80	33.1	30.9
	R1	
10	532.3	85.4
20	300.9	99.4
40	107.1	67.2
80	35.3	48.1
	R2	
10	870.1	146.1
20	431.1	131.9
40	103.4	66.0
80	16.8	25.9
	R3	
10	395.6	63.1
20	200.3	64.8
40	74.4	52.3
80	8.1	9.8
	R4	
10	1,043.5	169.1
20	369.1	118.6
40	59.9	44.2
80	21.5	19.2
	Mean	
10	652.9	107.4
20	283.4	91.8
40	82.5	55.4
80	22.2	27.0

Table A4

The average number of reinforcements in each 40-min session (rft/sess = reinforcements per session) and the corrected reinforcement and response rates on VR schedules for Green et al.'s (1982) 4 pigeons. Because Green et al. did not report postreinforcement pauses, each bird's response rates (rsp/min = responses per minute) and obtained reinforcement rates (rft/hr = reinforcements per hour) were corrected for obligatory pausing using a 0.91-s pause, which is the median of the smallest mean postreinforcement pauses from Brandauer's (1958) experiment. Mean data across the 4 birds are also listed in the table. The quantity, $\bar{n}$ represents the reciprocal of the slope of the linear feedback function (Equation 3) and the mean ratio requirement of the VR schedules. All quantities in the table were calculated from unrounded data.

$\bar{n}$	rft/sess	rft/hr	rsp/min
	P47		
12.5	362.9	631.2	131.6
25	154.7	246.5	102.9
50	85.7	132.9	110.8
100	47.2	72.1	119.1
200	9.4	14.2	45.7
400	1.4	2.1	13.0
	P48		
12.5	331.2	568.1	118.5
25	248.7	411.8	171.6
50	133.4	210.8	176.2
100	57.2	87.6	146.6
200	24.5	37.1	124.2
400	10.3	15.5	103.5
	P49		
12.5	316.2	538.9	111.7
25	204.7	332.8	138.6
50	111.1	174.0	144.7
100	56.9	87.2	145.6
200	25.3	38.3	128.2
400	8.4	12.6	76.5
	P50		
12.5	330.9	567.5	118.5
25	191.4	309.5	128.8
50	107.5	168.1	140.1
100	48.2	73.6	122.1
200	24.2	36.6	120.4
400	9.2	13.8	90.0
	Mean		
12.5	335.3	576.4	120.1
25	199.8	325.1	135.5
50	109.4	171.4	143.0
100	52.4	80.1	133.4
200	20.8	31.5	104.6
400	7.3	11.0	70.8

APPENDIX B

Derivation of Equation 4

Equation 4 is an algebraic rearrangement of Equation 2, which is

$$R_{\text{out}} = \left\{ \ln\left[1 + \frac{P_B}{\gamma P_R}\left(e^{1/R_{\text{in}}+w} - 1\right)\right] - w^* \right\}^{-1}. \tag{2}$$

Inverting, adding w^* to, and exponentiating both sides of this equation produces

$$e^{1/R_{\text{out}}+w^*} = 1 + \frac{P_B}{\gamma P_R}\left(e^{1/R_{\text{in}}+w} - 1\right),$$

and

$$e^{1/R_{\text{out}}}e^{w^*} = 1 + \frac{P_B}{\gamma P_R}\left(e^{1/R_{\text{in}}}\right)e^{w} - \frac{P_B}{\gamma P_R}.$$

Multiplying both sides by e^{-w^*},

$$e^{1/R_{\text{out}}} = e^{-w^*} + \frac{P_B e^{-w^*}}{\gamma P_R e^{-w}}\left(e^{1/R_{\text{in}}}\right) - \frac{P_B e^{-w^*}}{\gamma P_R},$$

and rearranging terms,

$$e^{1/R_{\text{out}}} = \frac{P_B e^{-w^*}}{\gamma P_R e^{-w}}\left(e^{1/R_{\text{in}}}\right) + e^{-w^*} - \frac{P_B e^{-w^*}}{\gamma P_R}.$$

Letting $(P_B e^{-w^*})/(\gamma P_R e^{-w}) = m$ and $e^{-w^*} - (P_B e^{-w^*})/(\gamma P_R) = b$, we have

$$e^{1/R_{\text{out}}} = m e^{1/R_{\text{in}}} + b.$$

Finally, taking the natural logarithm of both sides of this expression, and then inverting, produces

$$R_{\text{out}} = \left\{ \ln\left[m e^{1/R_{\text{in}}} + b \right] \right\}^{-1}, \tag{4}$$

which is Equation 4.

The Rate Equation's Point of Inflection and Differential Properties

The first two derivatives of Equation 4, the rate equation, are

$$R_{\text{out}}' = \frac{m e^{1/R_{\text{in}}}}{u(\ln u)^2 R_{\text{in}}^2}, \tag{B1}$$

and

$$R_{\text{out}}'' = \frac{m^2 e^{2/R_{\text{in}}}(2 + \ln u) - (1 + 2R_{\text{in}}) u m e^{1/R_{\text{in}}} \ln u}{u^2 (\ln u)^3 R_{\text{in}}^4}, \tag{B2}$$

where $u = m e^{1/R_{\text{in}}} + b$, and m and b are defined as for Equation 4.

As noted in the text, the rate equation has a point of inflection at a very low reinforcement rate (McDowell, 1979). At this point, the equation's curvature changes from concave upward to concave downward. The existence of the point of inflection is confirmed by the behavior of the rate equation's first two derivatives. Up to the point of inflection, the first derivative (or slope) of the rate equation is positive and increasing. At the point of inflection the first derivative is maximal, and beyond it the first derivative is positive and decreasing. The second derivative of the rate equation, which is the slope of the first derivative, is positive up to the rate equation's point of inflection. The second derivative passes through zero at the point of inflection (because the first derivative's slope is zero at its maximum), and then it becomes negative. The behavior of the second derivative allows one to calculate the rate equation's point of inflection. The x-coordinate of the point of inflection is obtained by setting the second derivative (Equation B2) equal to zero which, after some rearrangement, gives

$$0 = me^{1/R_{\text{in}}}(2 + \ln u) - (1 + 2R_{\text{in}})u \ln u. \tag{B3}$$

The solution of this equation for R_{in} gives the x-coordinate of the point of inflection. Equation B3 cannot be solved analytically, but for specific values of m and b it can be solved by a numerical procedure such as Newton's method (e.g., Heading, 1970). The y-coordinate of the point of inflection is obtained by substituting the value of R_{in} obtained from Equation B3 into Equation 4.

APPENDIX C

Parameter Estimates

Estimates of the parameters, m and b, in Equations 4 and 5 were obtained for each set of data from Brandauer (1958), Green et al. (1982), Mazur (1983), and McDowell and Wixted (1986) by fitting either Equation 4 or a linear approximation of Equation 5. For all experiments except those of McDowell and Wixted, response rates were corrected for obligatory pausing as explained in the text.

In the fits of Equation 4, "scheduled" reinforcement rates were used as the independent variable. These rates were calculated from Equation 3 by setting $\bar{n}$ equal to its scheduled value. (Notice that, except for Table A2, the tables in Appendix A list obtained rather than scheduled reinforcement rates.) Equation 4 was fitted by an iterative least-squares method like that described by McDowell and Kessel (1979).

In the fits of Equation 5, an approximate function form was used. Equation 5 can be fitted directly by iteratively adjusting the values of m and b with respect to a least-squares criterion. Every time the parameter values are adjusted, Equation 5 must be solved numerically for the predicted response rate at each $\bar{n}$. The differences between the predicted response rates and the corresponding observed response rates are then squared and summed to obtain the residual mean square. A simpler method of fitting Equation 5 was used here. This method entails first writing the equation in a form that permits analytic solution. Equation 5 is

$$R_{\text{out}} = \left\{\ln\left[me^{\bar{n}/R_{\text{out}}} + b\right]\right\}^{-1}, \tag{5}$$

where m and b are defined as in the text. Assuming that response and reinforcement rates in Equation 4 are expressed in the conventional units of responses per minute and reinforcements per hour, the units on $\bar{n}/R_{\text{OUT}}$ in Equation 5 must be converted to hours per reinforcement to produce identical units in the two equations. This is accomplished by inserting the conversion factor, 60 min/hr, into Equation 5 as follows:

$$R_{\text{out}} = \left\{\ln\left[me^{\bar{n}/(60R_{\text{out}})} + b\right]\right\}^{-1}.$$

The parameters m and b are directly interchangeable between Equation 4 and this version of Equation 5. Inverting and exponentiating both sides of the above equation produces

$$e^{1/R_{\text{out}}} = me^{\bar{n}/(60R_{\text{out}})} + b. \tag{C1}$$

This expression can be made more tractable by considering the series expansion of e^x, which is

$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots + x^r/r! + \ldots$$

(Heading, 1970). When x is small the second and higher order terms of the series contribute relatively little to its sum. Hence, when x is small,

$$e^x \cong 1 + x, \tag{C2}$$

(cf. McDowell, 1980, 1987). Evidently, for the variables in Equation C1, this approximation is best when $\bar{n}$ is small and response rate is large, and it becomes worse as $\bar{n}$ increases and response rate decreases. Substituting the appropriate versions of Equation C2 into both sides of Equation C1 produces

$$1 + 1/R_{\text{out}} = m[1 + \bar{n}/(60R_{\text{out}})] + b.$$

The advantage of this form of Equation 5 is that it can be solved for R_{out}:

$$R_{\text{out}} = \frac{-m}{60(m + b - 1)}\bar{n} + \frac{1}{m + b - 1}. \tag{C3}$$

Finally, letting

$$M = \frac{-m}{60(m + b - 1)}$$

and

$$B = \frac{1}{m + b - 1}, \tag{C4}$$

Equation C3 can be written

$$R_{\text{out}} = M\bar{n} + B. \tag{C5}$$

This equation, which is an approximation of Equation 5, is a line with a negative slope. It can be fitted by ordinary linear regression methods. Equations C4 show that the regression coefficients, M and B, are related to the parameters, m and b, as follows:

$$m = -60M/B \quad \text{and} \quad b = (60M + 1)/B + 1. \tag{C6}$$

Thus, estimates of m and b in Equation 5 can be obtained by fitting the linear form, Equation C5.

The methods of fitting Equations 4 and 5 described above are just two different ways of estimating the parameters m and b. Both methods were used for each set of data from Brandauer (1958), Green et al. (1982), Mazur (1983), and McDowell and Wixted (1986). The parameter estimates from the method that produced the better joint fit of Equations 4 and 5 are listed in Table C1. The equation from which the estimates were obtained is also listed (LA5 = linear approximation of Equation 5). The values of m and b in Table C1 were used in all subsequent calculations. It may be worth emphasizing that for a given set of data the values of m and b listed in Table C1 were used in both Equations 4 and 5.

Proportion of Variance Accounted For

Using the parameter values listed in Table C1, the proportions of variance accounted for (pVAF) by Equations 4 and 5 were calculated by the method described by McDowell (1981). These proportions are listed in the last two

Table C1

Equation fitted (LA5 = linear approximation of Equation 5), and estimated values of *m* and *b* in Equations 4 and 5, for data sets from McDowell and Wixted (1986; VI+ = VI plus linear feedback), Brandauer (1958), Mazur (1983), and Green et al. (1982). Values of *m* and *b* were calculated from unrounded data and are reported to nine decimal places.

Subject	Equation fitted	*m*	*b*
	McDowell and Wixted, 1986		
H31 (VR)	4	0.016 940 430	0.991 353 600
H31 (VI+)	4	0.011 507 300	0.996 639 900
H32 (VR)	4	0.024 045 790	0.984 236 300
H32 (VI+)	4	0.015 871 140	0.992 266 500
H36 (VR)	LA5	0.180 351 060	0.827 073 300
H36 (VI+)	LA5	0.175 398 490	0.831 420 500
H37 (VR)	LA5	0.205 867 250	0.800 234 900
H37 (VI+)	LA5	0.157 821 680	0.848 732 900
	Brandauer, 1958		
P14	LA5	0.070 284 377	0.934 430 000
P15	LA5	0.037 345 907	0.966 091 400
P17	4	0.069 887 040	0.934 990 000
Mean data	LA5	0.054 309 156	0.949 918 400
	Mazur, 1983		
M1	LA5	0.660 682 690	0.353 947 600
M2	LA5	0.526 551 570	0.481 244 500
M3	LA5	0.532 644 310	0.477 795 400
R1	LA5	0.393 387 540	0.616 664 000
R2	4	0.827 445 500	0.178 360 900
R3	LA5	0.623 308 480	0.389 542 200
R4	LA5	0.740 486 764	0.265 631 431
Mean data	LA5	0.603 857 510	0.404 984 600
	Green et al. 1982		
P47	4	0.145 033 470	0.862 782 800
P48	4	0.050 254 520	0.956 103 800
P49	LA5	0.057 301 473	0.949 738 900
P50	4	0.040 392 080	0.967 215 000
Mean data	LA5	0.071 250 501	0.935 910 900

columns of Table 1. In the case of Equation 5, the residuals were based on predicted response rates calculated by numerically solving Equation 5 at each $\bar{n}$. For reasons explained in the text, the larger of the two solutions of Equation 5 was used as the predicted response rate. For 3 of the 25 data sets, the largest $\bar{n}$ exceeded the boundary $\bar{n}$ entailed by Equation 5 and, consequently, a residual could not be calculated at that $\bar{n}$. In these three cases, which are identified in Table 1, the data point at the largest $\bar{n}$ was omitted. It is important to recognize that the pVAFs in the last column of Table 1 are the proportions of variance accounted for by Equation 5, not by its linear approximation (Equation C5), even though the latter equation may have been used to estimate the parameters *m* and *b*.

Point of Inflection

Using the parameter values listed in Table C1, Equation 4's point of inflection was calculated by the method described in Appendix B. The coordinates of the point of inflection for each data set are listed in Table 1.

Boundary VR

To determine the boundary VR, it is necessary to find the unique feedback function that is tangent to the rate equation. Referring to the top panel of Figure 5, it is clear that the slope of the tangent feedback function (in Figure 5's coordinates) is equal to the first derivative, or slope, of the rate equation at the point of tangency. Consequently, *at the point of tangency,*

$$\frac{R_{\text{out}}}{R_{\text{in}}} = \frac{dR_{\text{out}}}{dR_{\text{in}}}, \tag{C7}$$

where R_{out} and R_{in} are the coordinates of the point of tangency, their ratio is the slope of the tangent feedback function (in Figure 5's coordinates), and $dR_{\text{out}}/dR_{\text{in}}$ is the first derivative of the rate equation at the point of tangency. To find the tangent feedback function, it is necessary to find the coordinates, R_{out} and R_{in}, that satisfy Equation C7. The tangent feedback function is completely determined by its slope (the ratio of these coordinates) and by its intercept (zero).

Substituting the first derivative of the rate equation, as given by Equation B1 (Appendix B), into Equation C7 gives

$$\frac{R_{\text{out}}}{R_{\text{in}}} = \frac{me^{1/R_{\text{in}}}}{u(\ln u)^2 R_{\text{in}}{}^2},$$

or

$$R_{\text{out}} = \frac{me^{1/R_{\text{in}}}}{u(\ln u)^2 R_{\text{in}}}, \tag{C8}$$

where $u = me^{1/R_{\text{in}}} + b$ and m and b are defined as for Equation 4. Equation C8 expresses R_{out} at the point of tangency as a function of R_{in} at that point. But at this (and at every other) point, R_{out} is also given by the rate equation,

$$R_{\text{out}} = \left\{ \ln \left[me^{1/R_{\text{in}}} + b \right] \right\}^{-1} \tag{4}$$

or, with m, b, and u defined as for Equation C8,

$$R_{\text{out}} = 1/\ln u.$$

Substituting this expression into Equation C8 reduces the number of unknowns to one:

$$\frac{1}{\ln u} = \frac{me^{1/R_{\text{in}}}}{u(\ln u)^2 R_{\text{in}}},$$

or

$$1 = \frac{me^{1/R_{\text{in}}}}{R_{\text{in}} u \ln u},$$

and

$$0 = 1 - \frac{me^{1/R_{\text{in}}}}{R_{\text{in}} u \ln u}. \tag{C9}$$

The only reinforcement rate that satisfies Equation C9 is the reinforcement rate at the point of tangency. This rate can be obtained, given known values of the parameters m and b, by numerically solving Equation C9 for R_{in}. The response rate at the point of tangency can be calculated from Equation 4, using the reinforcement rate obtained from Equation C9 and the known values of m and b. The ratio of the reinforcement and response rates obtained in this way is the slope of the linear feedback function (Equation 3) that is tangent to the rate equation. Evidently, the quotient of the coordinates of the point of tangency, $R_{\text{out}}/R_{\text{in}}$ ($\times 60$ min/hr if the units on the two rates are responses per minute and reinforcements per hour, respectively), is the mean ratio requirement of the boundary VR schedule. Using the parameter values listed in Table C1, the boundary $\bar{n}$ and the predicted response rate at that $\bar{n}$ (which is just R_{out} at the point of tangency) were calculated for each set of data from Brandauer (1958), Green et al. (1982), Mazur (1983), and McDowell and Wixted (1986) and are listed in Table 1.

APPENDIX D

If data are available from a series of VI schedules and from a series of concurrent VR VI schedules, the linear system theory can be used to calculate the properties of responding on single-alternative VR schedules. The parameters m and b in Equation 4 can be estimated from the VI data, and the bias parameter in Equation 6 can be estimated from the concurrent VR VI data. The problem is to show how these estimates can be used to calculate m and b in Equations 4 and 5 for the VR case. In the following discussion, m and b for VI responding will be represented by m_{VI} and b_{VI}, m and b for VR responding will be represented by m_{VR} and b_{VR}, and P_B/P_R for the VI and VR cases will be represented by $P_{\text{VI}B}/P_{\text{VI}R}$ and $P_{\text{VR}B}/P_{\text{VR}R}$. In this notation the bias parameter in Equation 6 becomes $(P_{\text{VR}B}/P_{\text{VR}R})/(P_{\text{VI}B}/P_{\text{VI}R})$, assuming that the VR schedule is arranged on the second alternative. In all calculations in this appendix, w and w^* (the widths of the reinforcement and response pulses) will be assumed to be negligible (i.e., $\simeq 0$; see McDowell, 1979, 1980, 1987; McDowell & Kessel, 1979).

As can be seen from Appendix B, when w and w^* are negligible, Equation 4 for the VI case entails the parameters

$$m_{\text{VI}} = (1/\gamma)(P_{\text{VI}B}/P_{\text{VI}R}) \tag{D1}$$

and

$$b_{\text{VI}} = 1 - (1/\gamma)(P_{\text{VI}B}/P_{\text{VI}R}). \tag{D2}$$

Similarly, when w and w^* are negligible, Equation 4 for the VR case entails the parameters

$$m_{\text{VR}} = (1/\gamma)(P_{\text{VR}B}/P_{\text{VR}R}) \tag{D3}$$

and

$$b_{\text{VR}} = 1 - (1/\gamma)(P_{\text{VR}B}/P_{\text{VR}R}). \tag{D4}$$

Let us first consider the relationship between m_{VI} and m_{VR}. Dividing Equations D1 and D3 by the appropriate P_B/P_R parameter, we have

$$\frac{m_{\text{VI}}}{(P_{\text{VI}B}/P_{\text{VI}R})} = 1/\gamma$$

and

$$\frac{m_{\mathrm{VR}}}{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})} = 1/\gamma.$$

It follows that

$$\frac{m_{\mathrm{VR}}}{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})} = \frac{m_{\mathrm{VI}}}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}$$

and

$$m_{\mathrm{VR}} = m_{\mathrm{VI}}\frac{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}\,. \tag{D5}$$

The value of m_{VR} can be calculated from this equation given known values of m_{VI} (obtained from VI data) and $(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})/(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})$ (obtained from concurrent VR VI data).

To find the relationship between b_{VI} and b_{VR}, Equations D2 and D4 are rearranged to yield

$$\frac{1 - b_{\mathrm{VI}}}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})} = 1/\gamma$$

and

$$\frac{1 - b_{\mathrm{VR}}}{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})} = 1/\gamma.$$

It follows that

$$\frac{1 - b_{\mathrm{VR}}}{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})} = \frac{1 - b_{\mathrm{VI}}}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}\,.$$

Solving for b_{VR} gives

$$b_{\mathrm{VR}} = 1 + \frac{(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})}{(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})}(b_{\mathrm{VI}} - 1). \tag{D6}$$

The value of b_{VR} can be calculated from this equation given known values of b_{VI} and $(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})/(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})$.

As explained in the text, Equations D5 and D6 permit complete quantitative prediction of responding on single-alternative VR schedules, given data from single-alternative VI schedules and from concurrent VR VI schedules. Actually, Equations D5 and D6 permit complete quantitative prediction of responding on any of the three types of schedule given data from the other two.

In estimating the parameter, $(P_{\mathrm{VR}B}/P_{\mathrm{VR}R})/(P_{\mathrm{VI}B}/P_{\mathrm{VI}R})$, it may be necessary to use a function form other than Equation 6. For example, Equation 6 cannot be used when the data show undermatching, which is responding that is nearer indifference than required by matching (Baum, 1974, 1979). McDowell (1980) showed that Equation 6 is a simplified version of

$$\frac{R_{1_{\mathrm{out}}}}{R_{2_{\mathrm{out}}}} = \frac{(P_{2B}/P_{2R})}{(P_{1B}/P_{1R})}\left(\frac{e^{1/R_{2\mathrm{in}}+w} - 1}{e^{1/R_{1\mathrm{in}}+w} - 1}\right), \tag{D7}$$

which has an additional parameter, w, that accommodates undermatching. Equation D7 is the appropriate function form to use when it is necessary to take undermatching into account.