Short Report

The Wickelgren Power Law and the Ebbinghaus Savings Function

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Wayne Wickelgren, who died on November 2, 2005, after a long battle with Lou Gehrig's disease, studied the time course of forgetting more assiduously and more effectively than anyone since Hermann Ebbinghaus. In a classic article, Wickelgren (1974) derived an equation that is remarkable in several respects, including in its ability to characterize the famous Ebbinghaus (1885/1913) savings function. Under typical conditions, Wickelgren's power law reduces to

$$m = \lambda(1 + \beta t)^{-\psi}, \quad (1)$$

where m is memory strength, and t is time (i.e., the retention interval). The equation has three parameters: λ is the state of long-term memory at $t = 0$ (i.e., the degree of learning), ψ is the rate of forgetting, and β is a scaling parameter.

Wixted (2004) showed that Equation 1 provides an accurate description of forgetting data that have been averaged over many subjects. It not only fits the data well in terms of the percentage of variance accounted for—an admittedly weak test—but also accurately predicts where future points will fall as the retention interval increases, which is a stronger test. Because of possible averaging artifacts in group data, an even stronger test would be to accurately predict the course of forgetting using data from individual subjects. A practical problem with that approach is that such data are usually quite noisy. However, the eight data points of the Ebbinghaus (1885/1913) savings function constitute a rare and notable exception. Previous work has shown that the Ebbinghaus data are reasonably well characterized by a two-parameter power function of the form

$$m = \theta t^{-\psi}, \quad (2)$$

which can be considered an approximation of Equation 1 (Anderson & Schooler, 1991; Wixted & Ebbesen, 1991). Although Equation 2 offers a much better fit of the savings function than other two-parameter candidates, it is undefined at $t = 0$, which is theoretically unsatisfying and limits the equation's practical utility (e.g., it cannot be used to estimate the degree of learning).

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The left panel of Figure 1 shows the Ebbinghaus savings function along with the least squares fit of the Wickelgren power law (Equation 1). Not immediately apparent is the fact that there are actually four successive fits shown in that graph. In the first, Equation 1 was fit only to the first five points of the savings function (up to 24 hr), but was projected out to 31 days and drawn through all eight points. In the second, Equation 1 was fit to the first six points and then projected out to 31 days. In the third, it was fit to the first seven points, and in the fourth, it was fit to all eight. Remarkably, the four successive fits appear to be a single curve (i.e., they literally fall atop one another). The inset graph shows the fit of the power law using the data from the first 24 hr only. Although those five points appear to be almost vertically arranged in the larger graph, the general form of forgetting over 24 hr is actually much like the general form of forgetting over 31 days.

The right panel of Figure 1 shows a similar series of fits using another candidate function—the exponential function—of the form

$$m = (a - c)e^{-bt} + c, \quad (3)$$

where a is the degree of learning, b is the rate of forgetting, and c is the asymptote (Rubin, Hinton, & Wenzel, 1999). The shape of the exponential function often mimics that of the power function (Wixted, 2004), but the two functions differ in one theoretically intriguing respect. Specifically, whereas Equation 1 assumes that the forgetting function descends toward an asymptote of zero, Equation 3 allows for the possibility that it descends toward an asymptote greater than zero. From Figure 1, it is clear that Equation 3 systematically errs by overestimating where the next point will fall as the retention interval increases, such that the estimated asymptote declines as each new retention interval is added (as might be expected if the true asymptote were zero). This result stands in sharp contrast to the fit of the power law, which projects the same course of forgetting for each fit. Thus, the Wickelgren power law and the Ebbinghaus savings function conspire to suggest that forgetting functions ultimately project to an asymptote of zero.

Equation 1 not only is remarkable in its descriptive and predictive accuracy, but also offers a unique practical advantage

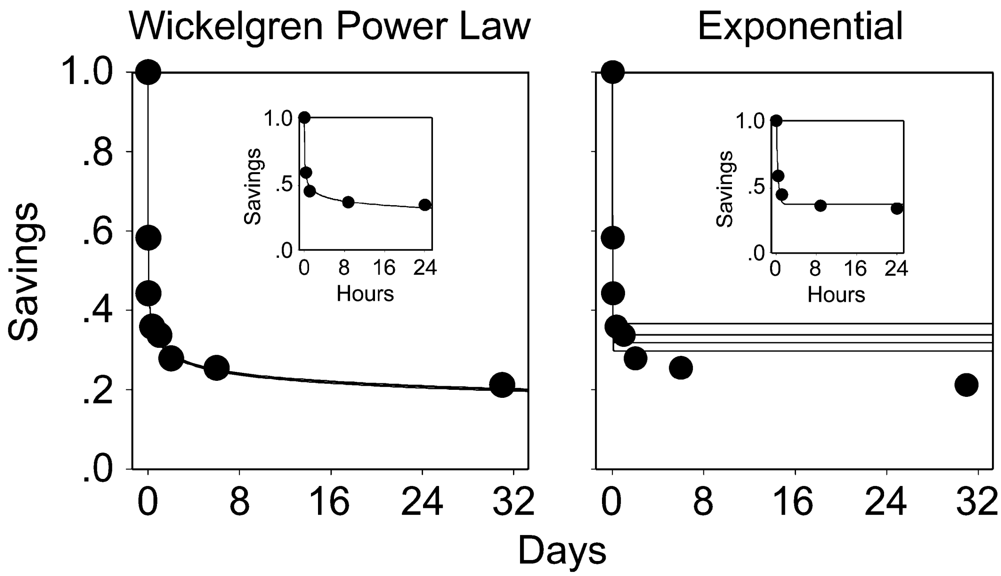

Fig. 1. Four successive least squares fits of Wickelgren's power law (Equation 1; left panel) and an exponential function (Equation 3; right panel) to the Ebbinghaus (1885/1913) savings data. The first fit involves the first five data points, the second involves the first six, the third involves the first seven, and the fourth involves all eight. Only a single curve is visually apparent in the left panel because the four curves fall atop one another. The inset graphs show the fits using only the first five points (through 24 hr). For these fits, both the power law and the exponential function account for 99% of the variance.

with regard to characterizing two properties of individual-subject forgetting functions that have long been of interest to memory researchers, namely, the *degree of learning* and the *rate of forgetting* (cf. White, 1985). Equation 1 has two parameters that correspond to those properties (λ and ψ, respectively). It also includes a scaling parameter (β), which is needed because time is measured in arbitrary units. However, it is reasonable to assume that this parameter remains constant across subjects and across conditions (i.e., subjects can be assumed to scale time in the same way), and doing so greatly reduces the number of parameters that need to be estimated. Specifically, each subject's data in a given condition can be fit by Equation 1, with λ and ψ free to vary across subjects (i.e., the degree of learning and the rate of forgetting are estimated for each subject), but with β constrained to be equal across subjects. For 30 subjects, this would mean estimating 60 parameters (the absolute minimum), plus 1 additional scaling parameter that is common to all subjects, for a total of 61 parameters. By contrast, fitting Equation 3 to 30 individual forgetting functions would require estimating 90 parameters because none of this equation's three parameters can be assumed to remain constant across subjects.

Wickelgren's contributions to the study of forgetting go well beyond the equation he proposed (e.g., Wixted, 2004). Still, in light of the ability of Equation 1 to so accurately characterize the venerable Ebbinghaus savings function, it seems appropriate to recognize the Wickelgren power law as an elegant contribution to the field—one that was never fully appreciated while Wickelgren was alive.

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(Received 4/26/06; Accepted 7/20/06;
Final materials received 8/15/06)